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SINGULAR PERTURBATIONS, ORDER REDUCTION, AND DECOUPLING OF LARG--ETC(U)
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(6) SINGULAR PERTURBATIONS, ORDER REDUCTION, AND DECOUPLING OF LARGE SCALE SYSTEMS.

by (9) Technical rept.

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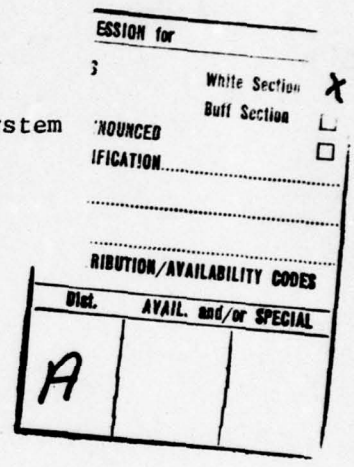
1. Introduction

Concepts from singular perturbations have recently suggested very useful new techniques for reduced order modeling, stability analysis, and synthesis of optimal controls (cf., e.g., Kokotović et al. (1976) and O'Malley (1978)). Their primary advantage is achieved by approximating solutions of systems of the form

$$(1) \qquad \begin{cases} \dot{u} = f(u,v,t,\varepsilon) \\ \varepsilon\dot{v} = g(u,v,t,\varepsilon) \end{cases}$$

with solutions of the lower order ("reduced" or "outer") system

$$(2) \qquad \begin{cases} \dot{U} = f(U,V,t,0) \\ 0 = g(U,V,t,0) \end{cases}$$



obtained by setting the small positive parameter ε equal to zero. In engineering contexts, analogous procedures have commonly (and necessarily) been used to lower the (often prohibitively high) dimensionality of complex models. These are often explained in terms of neglecting fast as opposed to slow dynamics (cf. Davison (1966), van Ness (1977), and Skira and De Hoff (1977)). Despite remarkable success, these schemes sometimes involve nonsensical use of asymptotic analysis, valid as $\varepsilon \to 0$, for $\varepsilon = 1$ (cf. Calise (1976)). Indeed, such success, so perilously based, requires more careful analysis. In the numerical methods literature such "singularly perturbed"

systems of differential equations are called stiff (cf. Willoughby (1974)). Singular perturbations theory has recently contributed to the analysis of such stiff equations (cf., e.g., Miranker (1975), Hemker (1977), Flaherty and O'Malley (1977), and Kreiss (1978)). In physical contexts, however, one is seldom presented with systems in the form (1). The fundamental question usually remains, viz: How does one numerically identify the small parameter(s) involved? This must be answered before one can effectively utilize the singular perturbations machinery, either directly or as an underpinning structure for a numerical approximations procedure.

As the simplest example of such problems, let us consider the linear constant coefficient system

$$\dot{x} = Ax \tag{3}$$

on a finite interval, say $0 \leq t \leq 1$, when the spectrum $\lambda(A)$ of A has two (or more) time scales (i.e., the eigenvalues of A cluster into two or more sets which are widely separated in magnitude). Specifically, we shall take x to be an n-vector with the spectrum of A being the disjoint sum

$$\lambda(A) = S \cup F \tag{4}$$

where

$$S = \{s_1, s_2, \ldots, s_{n_1}\} \quad \text{and} \quad |s_i| \leq |s_{i+1}| \quad \text{for each } i,$$

$$F = \{f_1, f_2, \ldots, f_{n_2}\} \quad \text{with} \quad |f_j| \leq |f_{j+1}| \quad \text{for each } j,$$

$n_1 + n_2 = n$ with $0 < n_1 < n$, and

$$|s_{n_1}| \ll |f_1|. \tag{5}$$

Thus, a small parameter appropriate for our analysis of (3) has been identified as

$$\varepsilon = |s_{n_1}/f_1| \ll 1. \tag{6}$$

More refined partitioning of $\lambda(A)$ (corresponding to several time scales) will often be useful as well, providing several small parameters of decreasing size. Our general approach may even be useful when ε is not so very small, as we've demonstrated numerically.

We shall also assume that

$$\text{Re } |f_j| \gg |s_{n_1}|, \quad j = 1, \ldots, n_1. \tag{7}$$

This will allow us to approximate the solutions of (3) by solutions of a lower order (n_1-dimensional) system, except in narrow endpoint boundary layers. We'd have to consider highly oscillatory solutions and use averaging methods if we omitted assumption (7) and allowed some eigenvalues f_j to be purely imaginary (or nearly so) (cf. Hoppensteadt and Miranker (1976) and Chow (1977), however).

2. Preliminary Mathematical Analysis

The solutions of (3) can, of course, be obtained by constructing the matrix exponential e^{At}, or $e^{At}K$ for any nonsingular matrix K. As

we'll find, however, its numerical computation is by no means straightforward (cf. Moler and van Loan (1978)). We shall present a particular way of approximating the solutions of (3) by its n_1 slow modes, adding endpoint corrections due to the n_2 fast modes as needed. To proceed, let us suppose x is decomposed into an n_1 and an n_2 dimensional subvector and A is subdivided in a compatible fashion, i.e.,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{8}$$

We shall transform x to an n-vector y with separated slow and fast subvectors y_1 and y_2. Thus, we set

$$x = \begin{bmatrix} I_{n_1} & -K \\ -L & I_{n_2} + LK \end{bmatrix} y \tag{9}$$

to obtain

$$\dot{y} = By = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} y \tag{10}$$

for

$$B_1 = A_{11} - A_{12}L \quad \text{and} \quad B_2 = A_{22} + LA_{12} \tag{11}$$

where L is chosen so that

$$\begin{cases} \lambda(B_1) = S = \{s_1, \ldots, s_{n_1}\} \\ \text{and} \end{cases} \tag{12}$$

$$\left\{ \lambda(B_2) = F = \{f_1, \ldots, f_{n_2}\}. \right. \tag{12}$$

We note that the transformation (9) is suggested by that commonly used for time-varying systems (cf., e.g., Harris (1973)). Setting $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we have the separate systems

$$\dot{y}_1 = B_1 y_1 \tag{13}$$

and

$$\dot{y}_2 = B_2 y_2. \tag{14}$$

Now the singular perturbations nature of (10), or equivalently (13)-(14), would be made even more obvious if we had multiplied (14) through by the small parameter $\|B_1\|/\|B_2\|$ and compared the result with (1).

We shall obtain the block diagonal system (10) in two steps. If we first take $K = 0$ in (9), we find that $\begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$ satisfies the decoupled upper block triangular system

$$\begin{pmatrix} \dot{x}_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} B_1 & A_{12} \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \tag{15}$$

provided the (non-square) decoupling matrix L satisfies the algebraic Riccati equation

$$LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0 \tag{16}$$

in addition to the eigenvalue placement requirements (11)-(12). (We note

that (16) is linear when $A_{12} = 0$. Then, however, A is already in block-triangular form, although not generally decoupled into slow and fast modes.) We next obtain the diagonal system (10) by requiring K to satisfy the linear equation

$$KB_2 - B_1K + A_{12} = 0. \tag{17}$$

This Liapunov equation will have a unique solution since B_1 and B_2 have no eigenvalues in common (cf., e.g., Bellman (1970)). An attractive feature of the nonsingular transformation (9) is that its inverse is simply given by

$$y = \begin{bmatrix} I_{n_1} + KL & K \\ L & I_{n_2} \end{bmatrix} x, \tag{18}$$

so we don't need to numerically perform a matrix inversion to solve (9) for y.

Presuming we can solve for L and K, we can treat the system (3) as the transformed system (10) and then separately integrate (13) for the slowly-varying vector y_1 and (14) for the rapidly-changing vector y_2. If m_1 of the (large) eigenvalues f_j of B_2 have negative real parts, there will be an m_1-dimensional manifold of solutions to (14) which decay rapidly to zero away from $t = 0$ and an $m_2 = n_2 - m_1$ dimensional manifold of rapidly growing solutions there. Alternatively, $e^{B_2(t-1)}$ has an m_2 dimensional column space of vectors which decay rapidly to zero within $[0,1)$, away from $t = 1$. Let us use the (not necessarily Jordan) decomposition

(19) $$B_2 = NDN^{-1}$$

where D has the block diagonal form

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

with D_1 and $-D_2$ being $m_1 \times m_1$ and $m_2 \times m_2$ dimensional stable matrices, and

$$N = [N_1 \quad N_2]$$

where N_1 is $n_2 \times m_1$ dimensional. Then

$$e^{B_2 t} N_1 = N_1 e^{D_1 t}$$

spans all solutions of (14) which decay away from $t = 0$, while

$$e^{B_2 (t-1)} N_2 = N_2 e^{D_2 (t-1)}$$

spans all solutions of (14) which decay away from $t = 1$. For the matrix

$$C = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N_1 + e^{-B_2} N_2 \end{bmatrix},$$

(20) $$Y = e^{Bt} C = \begin{bmatrix} e^{B_1 t} & 0 \\ 0 & e^{B_2 t} N_1 + e^{B_2 (t-1)} N_2 \end{bmatrix}$$

will be a bounded fundamental matrix for the system (10). Our use of the $e^{B_1 t}$ notation is harmless, since any standard numerical code or explicit formulas for the matrix exponential will serve well to produce this slowly-varying solution to the matrix version of (13). Unless B_2 or $-B_2$ is stable, however, $e^{B_2 t}$ is very difficult to compute, since it contains both rapidly growing and rapidly decaying modes (though no slow modes). Less difficulty is involved in separately computing its fast initial and terminal transients $e^{B_2 t} N_1$ and $e^{B_2 (t-1)} N_2$. These correspond to integrating (14) on the appropriate m_1 or m_2 dimensional manifold. If we wished to obtain a bounded solution to the initial value problem for (3), to obtain a bounded solution on $[0,1]$ as $\varepsilon \to 0$, we'd need $m_1 = n_2$ or we'd have to restrict the initial vector $x(0)$ to a lower dimensional manifold. We might even consider such problems on $t \geq 0$ if A were stable. Two-point problems, of course, need not have a solution, but we've at least outlined a framework in which we might numerically consider the question. These considerations, of course, parallel those always encountered in using fundamental matrices (cf., e.g., Coppel (1965) and Cole (1968)).

It is convenient to decompose the fundamental matrix $e^{Bt}C$ for (10) as

$$Y = e^{Bt}C = (Y_s \quad Y_{f0} \quad Y_{f1}) \tag{21}$$

with n_1 slow modes

$$Y_s = \begin{pmatrix} Y_{s1} \\ Y_{s2} \end{pmatrix} = \begin{pmatrix} e^{B_1 t} \\ 0 \end{pmatrix}, \tag{22}$$

m_1 fast-decaying initial modes

$$Y_{f0} = \begin{pmatrix} Y_{f01} \\ Y_{f02} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{B_2 t} N_1 \end{pmatrix}, \tag{23}$$

and m_2 fast-decaying terminal modes

$$Y_{f1} = \begin{pmatrix} Y_{f11} \\ Y_{f12} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{B_2 (t-1)} N_2 \end{pmatrix}. \tag{24}$$

The decomposition (21) implies a corresponding decomposition for a bounded fundamental matrix $e^{At}C$ for our original system (3), i.e.,

$$X = e^{At}\tilde{C} \equiv (X_s \quad X_{f0} \quad X_{f1}) \tag{25}$$

with n_1 slow modes

$$X_s = \begin{pmatrix} X_{s1} \\ X_{s2} \end{pmatrix} = \begin{pmatrix} I_{n_1} & -K \\ -L & I_{n_2} + LK \end{pmatrix} Y_s = \begin{pmatrix} I_{n_1} \\ -L \end{pmatrix} e^{B_1 t} \tag{26}$$

and n_2 fast modes

$$X_{f0} = \begin{pmatrix} X_{f01} \\ X_{f02} \end{pmatrix} = \begin{pmatrix} -K \\ I_{n_2} + LK \end{pmatrix} e^{B_2 t} N_1 \tag{27}$$

and

$$(28) \qquad X_{f1} = \begin{pmatrix} X_{f11} \\ X_{f12} \end{pmatrix} = \begin{pmatrix} -K \\ I_{n_2} + LK \end{pmatrix} e^{B_2(t-1)} N_2.$$

(If $m_2 = 0$, e.g., X_{f1} is omitted in (25).) Within (0,1), then, all solutions to (3) are nicely approximated in the column space of the reduced order $n \times n_1$ matrix $X_s(t)$. In the initial "boundary layer," i.e., in a narrow interval near $t = 0$, X_s is nearly constant, so solutions are well approximated by the span of the $n \times (n_1 + m_1)$ matrix $(X_s(0) \;\; X_{f0})$, while $(X_s(1) \;\; X_{f1})$ is appropriate near $t = 1$. If we're not concerned with the fast endpoint transients, we'd simply use the reduced (or "outer") approximation $X_s(t)$ everywhere. Since $X_s(0)$ has rank $n_1 < n$, while x may be determined by n linearly independent endconditions, we realize that this approximation is usually not correct at $t = 0$ and 1. The approximation of x by a vector multiple of X_s, however, improves as the ratio ε decreases (i.e., the stiffer our system (3), the more negligible are the fast modes X_{f0} and X_{f1} within (0,1)). We note that the separation of growing and decaying modes is common in time-varying stability theory and is formalized in the concept of an exponential dichotomy (cf. Coppel (1978)). It has previously been used by Ferguson (1975) for the numerical solution of singular perturbation problems. Finally, we emphasize the numerical importance of decoupling the slow part and the fast initial and terminal transients. Large integration steps can be used throughout to calculate X_s, while an accurate integration of X_{f0} and X_{f1} (with much smaller stepsizes) can be obtained, if needed, on the appropriate short t-intervals. We intend to primarily emphasize the numerical aspects involved with computing the reduced order model X_s. Readers should note that this primarily involves the decoupling matrix L,

although the matrix K is needed to identify the endconditions appropriate for X_s.

3. The Decoupling Matrices L and K

In order to utilize the structure we've developed, we must be able to find an $n_2 \times n_1$ matrix solution L of the algebraic Riccati equation (16) satisfying the eigenvalue placement conditions (11)-(12) as well. We note that the well-studied case when A is a Hamiltonian matrix is very important in applications (then $A_{11} = -A_{22}'$, $A_{12} = A_{12}'$, $A_{21} = A_{21}'$ (with the prime denoting transposition), and the solution of (16) is symmetric). Under assumptions of stabilizability and detectability, (16) has a unique positive semi-definite solution (cf. Kucera (1973)). An analogous approach is required for our problem. Specifically, let us suppose that A has the decomposition

$$A = MJM^{-1} \tag{29}$$

where

$$J = \begin{bmatrix} J_s & 0 \\ 0 & J_f \end{bmatrix}, \tag{30}$$

with the blocks J_s and J_f satisfying

$$\lambda(J_s) = \lambda(B_1) = S, \quad \lambda(J_f) = \lambda(B_2) = F, \tag{31}$$

and the matrix M having the compatible partitioning

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \tag{32}$$

We could take M to be a modal matrix so that J is in Jordan form. Then the columns of M would be right eigenvectors and generalized right eigenvectors of A corresponding to the diagonal entries of J, while the rows of M^{-1} would be corresponding (sometimes generalized) left eigenvectors of A (cf., e.g., Hirsch and Smale (1974)). It's not necessary, however, to be so explicit. All that is needed is the slow-fast block decomposition of J provided by (30)-(31). Indeed, the Jordan form is not particularly well suited to numerical computation (cf. Golub and Wilkinson (1976)).

Since $e^{At}M = Me^{Jt}$ is a fundamental matrix for (3), it follows that the n_1-dimensional space of slowly-varying solutions of (3) coincides with the column span of

$$M \begin{bmatrix} e^{J_s t} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} e^{J_s t}, \tag{33}$$

$e^{J_f t}$ being fast-varying. Comparing with the slowly-varying matrix X_s of (26), it follows that $\begin{pmatrix} I_{n_1} \\ -L \end{pmatrix}$ and $\begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix}$ must span the same space. Therefore, if we have a solution L to (16), (11), and (12), M_{11} will be invertible and we must have

$$L = -M_{21}M_{11}^{-1}. \tag{34}$$

Conversely, if $LM_{11} = -M_{21}$ for M_{11} invertible, multiplying out

$$\begin{bmatrix} I_{n_1} & 0 \\ L & I_{n_2} \end{bmatrix} A \begin{bmatrix} I_{n_1} & 0 \\ -L & I_{n_2} \end{bmatrix}$$

and using the decomposition (29) for A produces the lower right entry $LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0$ (after considerable manipulation). Further, if

$$M^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \tag{35}$$

we'll have $Q_{22} = (M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1}$ and (34) will hold if and only if

$$L = Q_{22}^{-1}Q_{21}. \tag{36}$$

The formulas (34) and (36) show that the matrix L we are seeking is unique, although M is far from unique and the quadratic equation (16) has many other solutions. Wu and Narasimhamurthi (1976), Narasimhamurthi and Wu (1977), and Anderson (1978) contain alternative and more detailed treatments of these results.

In order to compute L, we first employ a standard eigenanalysis library routine to obtain approximate eigenvalues of the matrix A. This allows us to select n_1, the number of moderate eigenvalues, and thereby determines the number $n_2 = n - n_1$ of eigenvalues with large real parts. We note that multiple and complex conjugate eigenvalues are naturally grouped together in either S or F.

If $n_1 \leq n/2$, we obtain n_1 approximate right (or generalized right) eigenvectors corresponding to the n_1 moderate eigenvalues, thereby forming an approximation $\begin{bmatrix} M_{11}^{\circ} \\ M_{21}^{\circ} \end{bmatrix}$ to the submatrix $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ of (32). (If the resulting matrix has complex entries, we eliminate them through

elementary column operations.) Then we solve the resulting real system

$$\tilde{L}_0 M_{11}^\circ = -M_{21}^\circ, \tag{37}$$

approximating (34), providing M_{11}° is nonsingular. An efficient method of obtaining an approximate solution L_0 to (37) is the "LU" factorization technique (cf., e.g., Stewart (1973)). If $n_1 > n/2$, it would be more efficient to seek $n_2 < n/2$ approximate left (or generalized left) eigenvectors of A and instead obtain an approximate solution to (36). In either case, we don't need all the eigenvectors of A. For all our computations to date, we have used a modal matrix for M, but we expect to not be so restrictive in later testing.

Thus far, we have used x_1 and x_2 as any n_1 and n_2 dimensional subvectors of x, requiring only that the matrix $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ of slow modes (necessarily of rank n_1) have M_{11} nonsingular. This may demand a reordering of the components of x. If we used an ordering for which M_{11} was nearly singular, the approximate solution L_0 would generally be of large norm and, quite likely, difficult to obtain accurately. A small approximate L_0 would, however, result if we kept physical coordinates known to be primarily slowly-varying in x_1, with coordinates dominated by fast boundary-layer variations being kept in x_2. Such a splitting of x would correspond roughly to the splitting $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ into slow and fast components. When x_1 and x_2 are weakly coupled like this, the need for the decoupling matrix L in (9) is lessened, so we would expect L to be small. Formally, then, it is natural to introduce a slow-mode coupling ratio

(38) $$\rho_s = \min (\|M_{21}\|/\|M_{11}\|)$$

where the minimization takes place over all possible reorderings of the rows of the matrix M. If we could obtain $M_{21} = 0$, x_1 and x_2 would be decoupled and we'd have $L = 0$ by (34). The practical significance of such reorderings can be seen in our numerical results and is stressed in the aircraft dynamics examples of Teneketzis and Sandell (1977), though it is certainly not necessary when the original M_{11} matrix is nonsingular.

An efficient method of improving the initial approximation L_0 to the decoupling matrix L can be obtained by using a successive approximations scheme on the algebraic Riccati equation (16) with L_0 as the first iterate. Using (11), we can rewrite (16) as

(39) $$L = B_2^{-1}(LB_1 + A_{21} - LA_{12}L) = B_2^{-1}(LA_{11} + A_{21}).$$

Thus, we naturally define the iterates

(40) $$L_{j+1} = (A_{22} + L_jA_{12})^{-1}(L_jA_{11} + A_{21}), \qquad j = 0, 1, \ldots,$$

expecting them to converge to the unique L desired. This iteration scheme features rapid convergence provided L_0 is close to L and ε is small enough (cf. Kleinman (1968) who discussed the analogous Hamiltonian problem). Indeed, a proof of convergence follows from a contraction mapping argument. If we introduce

(41) $$D_j = L_{j+1} - L_j,$$

we find that (40) is equivalent to

$$D_j = (A_{22} + L_j A_{12})^{-1} R_j, \qquad j = 0, 1, \ldots, \tag{42}$$

for the residual

$$R_j = L_j A_{11} - A_{22} L_j - L_j A_{12} L_j + A_{21}. \tag{43}$$

The procedure terminates when one finds a suitably small R_j. (In particular, if R_0 is judged small enough, we wouldn't begin the iteration scheme (40) or (42).) Further manipulating (42) and (43) implies that

$$D_{j+1} = (A_{22} + L_j A_{12})^{-1} D_j (A_{11} - A_{12} L_{j+1}). \tag{44}$$

We note that $A_{22} + L_j A_{12}$ will be a good approximation to B_2, if L_j is a good approximation to L. Then, the inverse used in (40)-(44) is legitimate. Likewise, $A_{11} - A_{12} L_{j+1}$ will then be a good approximation to B_1, and since $\|B_1\| / \|B_2\| = O(\varepsilon)$, (44) implies that our scheme ultimately has a rate of convergence of order ε. A closely related approach is presented in Kokotovic (1975) and Chow and Kokotovic (1976). Instead of (37), they used the initial iterate $L_0 = A_{22}^{-1} A_{21}$, appropriate if A_{22} is nonsingular and if the system is sufficiently decoupled so that L is small and thereby easy to compute (cf. (40)). Thus, their convergence criterion is unnecessarily restrictive. Numerical experiments indicate that our scheme is successful even when a rather crude solution to (37) is used for L_0, and when ε is not too small.

After numerically obtaining the decoupling matrix L, we need to solve the linear equation (17) for the matrix K. Rewriting it as

(45) $$K = (B_1K + A_{12})B_2^{-1}$$

suggests the iteration scheme

(46) $$K_{j+1} = (B_1K_j + A_{12})B_2^{-1}, \quad K_0 = 0, \quad j = 0, 1, 2, \ldots,$$

converging to its unique solution. A straightforward argument implies a rate of convergence of order ε. Here, unlike for (40), the initial iterate is not crucial. Alternatively, one can use the representations (32) and (35) to show that

(47) $$K = -M_{12}Q_{22} = M_{11}Q_{12}$$

(cf. Anderson (1978)), though this requires more information about the modal decomposition of A than (46).

4. Implementation

We propose to obtain a reduced order matrix approximation for two-time-scale systems $\dot{x} = Ax$ via the following numerical algorithm:

1) Obtain approximate eigenvalues of A and order then by increasing moduli into sets S and F satisfying (7). Note that different choices are possible for the number n_1 of slow eigenvalues for any given matrix A.

2) Determine an $n \times n_1$ dimensional real matrix $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ whose columns span the right eigenspace of A corresponding to the slow eigenvalues. (If $n_1 > n/2$, an alternative based on left eigenvectors is suggested (cf. (36)).

3) (optional) Reorder the variables x so that $\|M_{21}\|/\|M_{11}\|$ is minimized.

4) Find an approximate solution L_0 of $L_0M_{11} = -M_{21}$ (or its alternate).

5) (optional) Improve the accuracy of the approximation L_j by iteration. (a) Stop if $\|R_j\|$ is less than a prescribed tolerance for $R_j = L_jA_{11} + A_{21} - L_jA_{12}L_j - A_{22}L_j$. (b) Set $L_{j+1} = L_j + (A_{22} + L_jA_{12})^{-1}R_j$ and return to (a) with $j = j + 1$.

6) The approximate reduced order fundamental matrix (appropriate within $(0,1)$) is

$$X_s^{(j)} = \begin{pmatrix} I_{n_1} \\ -L_j \end{pmatrix} e^{(A_{11}-A_{12}L_j)t}$$

where j is zero or the last index used in 5).

This algorithm has been applied to a number of physical systems with order n ranging up to 32. Two different programs were used for the eigenanalysis of steps 1) and 2). They are the EISPACK subroutines, which are available without cost from the Applied Mathematics Division of the Argonne National Laboratory (cf. Smith et al. (1976)), and EIGRF, which is part of the proprietary subroutines sold by the International Mathematics and Statistics Library (IMSL) of Houston. The procedure was implemented on the 64 bit CYBER 175 and the 36 bit DEC 10 computers at the University of Arizona Computing Center.

Example 1. This fourth order model of F-8 aircraft longitudinal dynamics (cf. Etkin (1972) and Teneketzis and Sandell (1977)) has slow eigenvalues $s_{1,2} = -0.0075 \pm i0.076$ and fast eigenvalues $f_{1,2} = -0.94 \pm i3.0$, corresponding to the small parameter $\varepsilon = |s_2/f_1| = 0.024$. The

physical variables involved are velocity variation x_1 (ft./sec.), flight path angle x_2 (rad.), angle of attack x_3 (rad.), and pitch rate x_4 (rad./sec.). The first two are primarily slow, while the latter two are predominantly fast. The four different orderings of the x coordinates with $n_1 = n_2 = 2$ show that the ratio $\|M_{21}\|/\|M_{11}\|$ and the corresponding $\|L\|$ are loosely related. The numbers are $(1.9 \times 10^{-4}, 1.1 \times 10^{-3})$, $(2.4 \times 10^{-3}, 1.3 \times 10^{3})$, $(5.3 \times 10^{3}, 6.3 \times 10^{3})$, $(4.2 \times 10^{2}, 2.0 \times 10^{4})$ so $\rho_s = 1.9 \times 10^{-4}$ for the ordering suggested above. The success of the iterative procedure is illustrated in Figure 1 where $\log \|R_j\|$ is plotted against j. We find that $\|R_{j+1}\|/\|R_j\| \approx \varepsilon$ until the (machine-dependent)

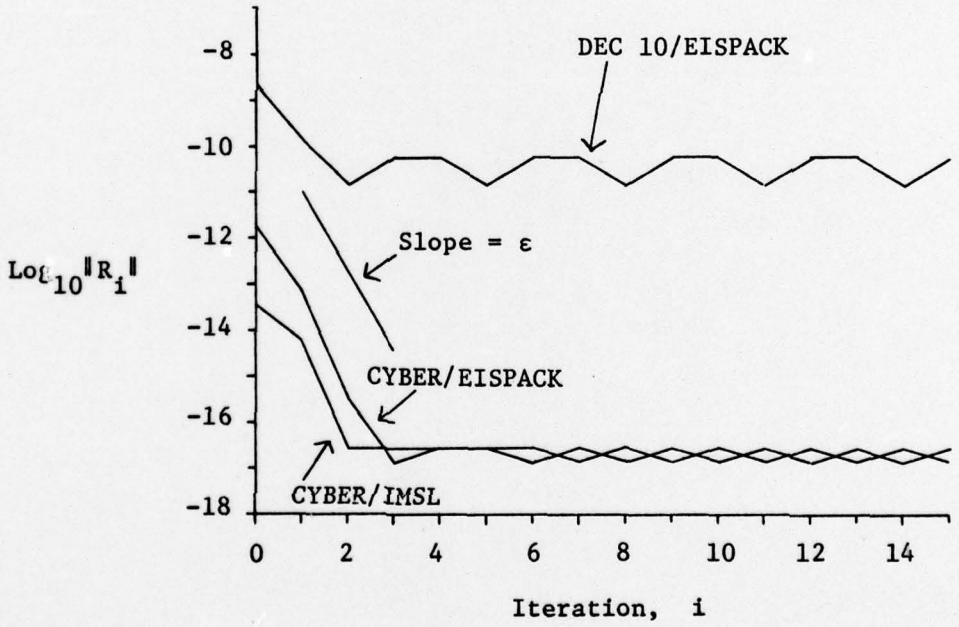


Figure 1.

The convergence of L_i for the F-8 aircraft longitudinal dynamics model.

maximum possible numerical accuracy is achieved. These results also show that the approximate matrix $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ obtained using IMSL was better than with EISPACK, since the resulting L_0 was more accurate. Further tests show, however, that considerable eigenvector inaccuracies may be tolerated with the algorithm still converging to the desired solution when iteration is used. Indeed, the algorithm converges whatever ordering is used for the x components. Most important, however, is the fact that trajectories for initial value problems are well approximated, outside an initial boundary layer region, by the reduced model. Using initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0)) = (100, 0, 0, 0.5)$, the aircraft response for the predominantly slow and fast variables x_1 and x_3 are shown in Figures 2 and 3, both for the full fourth order model and the reduced second order model. After the first few seconds the responses are indistinguishable, so they are not pictured.

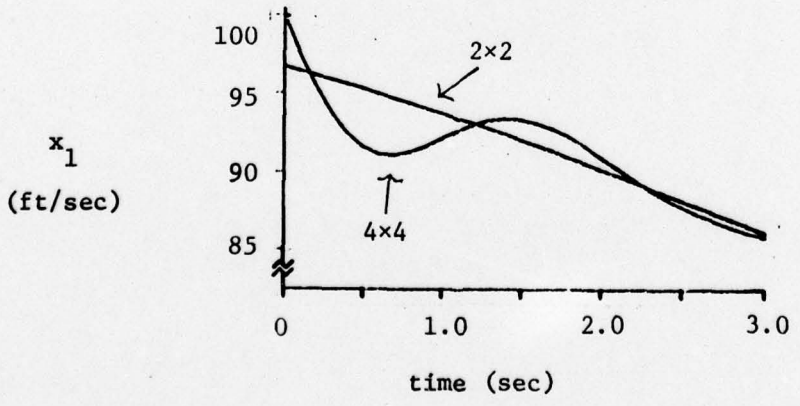


Figure 2.

Velocity (slow variable) vs. time.

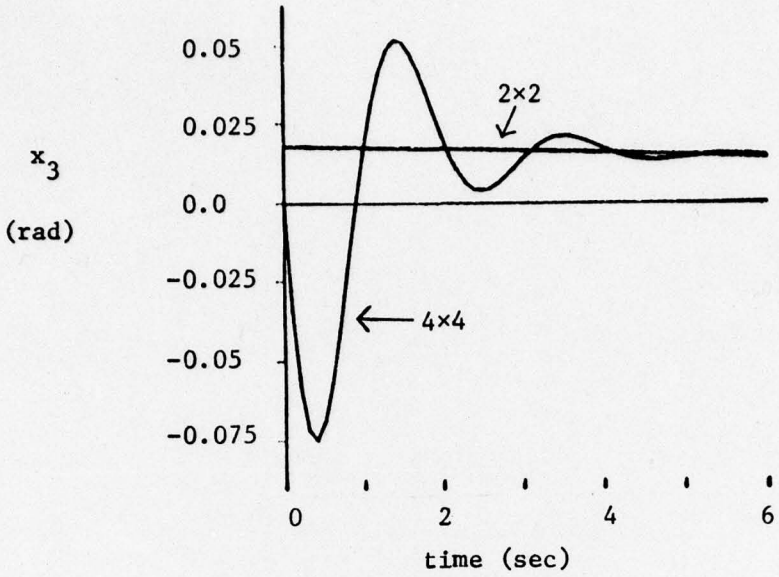


Figure 3.

Angle-of-attack (fast variable) vs. time.

Example 2. This 16^{th} order model of an F-100 turbofan engine was used as the theme problem for the recent International Forum on Alternatives for Multivariable Control (cf. Sain (1977)). Two of the x coordinates are shaft speeds, three are pressures, and eleven are temperatures. The eigenvalues (ordered in magnitude) are -0.648, -1.906, -2.619, $-6.715 \pm i1.312$, $-17.8 \pm i4.78$, -18.6, $-21.3 \pm i0.822$, -38.7, -47.1, -50.7, -59.2, -175.7, and -577.0. If we take $n_1 = 15$, 5, and 3, we get ε values 0.304, 0.371, and 0.383, respectively. The choice $n_1 = 15$ is ruled out because the reduction in dimensionality isn't significant. Even though the remaining ε's aren't very small, our computational algorithm was successful in producing appropriate decoupling matrices L. Results are contained in Figure 4 for $n_1 = 3$.

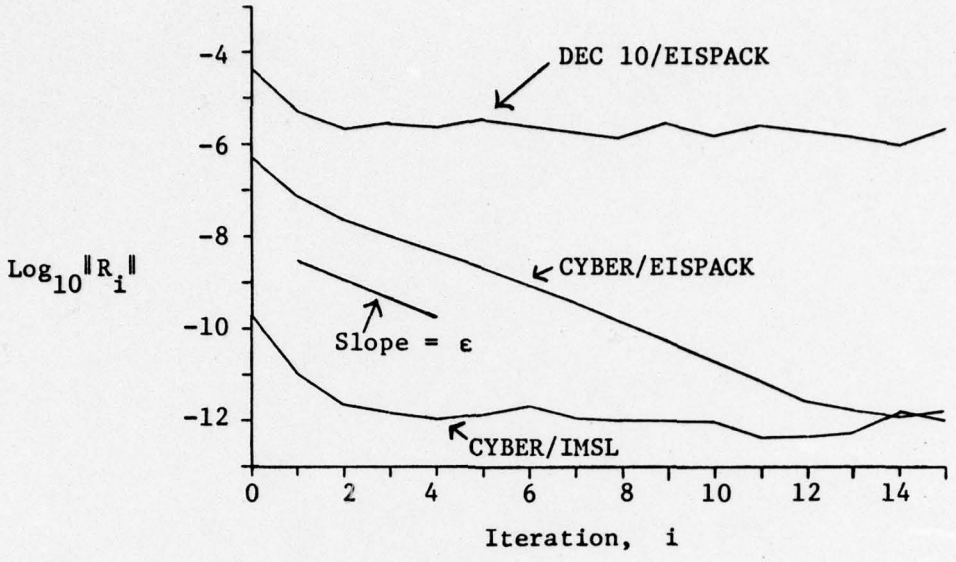


Figure 4.

The convergence of L_i for decoupling the three slowest modes for the F-100 turbofan engine.

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Singular perturbation methods have recently provided new techniques for optimal control and stability problems. Their primary advantage is the order reduction achieved when small parameters are set equal to zero. We present an approach for identifying small parameters by separating slow and fast modes. A computational procedure is developed for simple linear systems and tested numerically. Connections to current engineering literature is stressed.

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